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Reflector synthesis for generalized far-fields

B S Westcott and A P Norris
Mathematics Department, University of Southampton, Southampton SO9 5NH, UK

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Abstract. The theoretical foundations of a proposed new method of reflector design are presented. The geometric-optics approximation is used to synthesise a reflector surface to produce a generalized far-field when illuminated by a point source. The far-field can have a prescribed variation in two variables and source blockage can be removed. The techniques of differential geometry are used to show that a nonlinear second-order partial differential equation, the Monge–Ampère equation, must be solved subject to nonlinear boundary conditions. Particular consideration is given to the special case of fields with even azimuthal symmetry when referred to spherical polar coordinates. The method is applicable to problems in optics, acoustics and microwave antenna design where shaped beams are required.

1. Introduction

A new method of designing reflector surfaces under the geometric-optics approximation has been introduced by the authors in a recent letter (Norris and Westcott 1974). The theoretical foundations of the method are elaborated in this paper.

The objective is to synthesise a reflector surface which is capable of producing a generalized far-field pattern when it is illuminated by a point source.

Subject to certain constraints on surface curvature this problem is shown, by the use of differential geometry, to be reduced to solving a nonlinear second-order partial differential equation.

In the past systematic methods in geometric design (see for example Silver 1949) have been restricted to a far-field variation in one variable, where the problem reduces to solving an ordinary differential equation. The present analysis extends the scope of geometric design to two variables thereby introducing a much greater flexibility in the realization of required far-fields.

In order to solve the partial differential equation a boundary condition is formulated which allows elliptic forms of the equation to be solved numerically. The design procedure can be automated and the method has been successfully programmed on a digital computer.

The method would seem to be generally useful in the optical and acoustic fields and particularly useful in dealing with the requirements of antennae in microwave systems, where a far-field with a prescribed functional variation over a specified solid angle in space is desirable.

The theoretical foundations are detailed in §§ 2 and 3 where for generality and indeed brevity the methods of tensor calculus are used to derive the relevant equations. The resultant analysis is then referred to spherical coordinates in § 4. Boundary conditions for the elliptic form of the partial differential equation are discussed in § 5 and some implications about existence and uniqueness of solutions in § 6. Finally some conclusions are presented in § 7. Numerical results will be presented in subsequent papers.

2. Mathematical formulation of the problem

The two main concepts used in geometrical design are the conservation of energy and the law of reflection. With reference to the geometry of figure 1 the energy conservation law may be written in the form

$$D = \frac{G}{I} = \left|\frac{\mathrm{d}\Omega'}{\mathrm{d}\Omega}\right|, \tag{1}$$

where G, I are the reflector far-field and incident power densities respectively and $\mathrm{d}\Omega'$, $\mathrm{d}\Omega$ are solid angles subtended respectively by the elementary incident and reflected ray cones indicated in the figure.

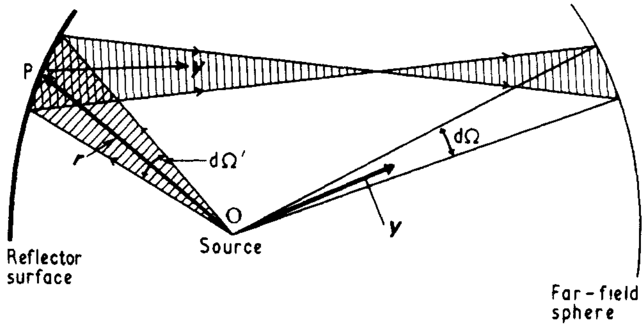

Figure 1. Geometry showing elementary incident and reflected ray cones.

It is noted that if the feed is isotropic then I is a constant and D is effectively the far-field pattern.

Consider a point P on a reflector surface with position vector $\boldsymbol{r}$ relative to source point O. The unit vectors defining incident ray direction $\boldsymbol{r}/r$ and reflected ray direction $\boldsymbol{y}$ at P are related by the law of reflection to a surface normal vector $\boldsymbol{n}$ where

$$\boldsymbol{n} = r\boldsymbol{y} - \boldsymbol{r}. \tag{2}$$

Of course the vector $\boldsymbol{n}$ is *not* a unit vector when given by (2).

It is now assumed that the direction $\boldsymbol{y}(u^1, u^2)$ is parametrized by two generalized coordinates u^1, u^2 on the unit sphere $|\boldsymbol{y}| = 1$ (say θ, ϕ in a spherical polar coordinate system with origin at 0).

The elementary solid angles are then

$$\mathrm{d}\Omega' = -\frac{1}{r^3}[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2]\,\mathrm{d}u^1\,\mathrm{d}u^2$$

$$\mathrm{d}\Omega = [\boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2]\,\mathrm{d}u^1\,\mathrm{d}u^2,$$

where

$$r_i = \frac{\partial r}{\partial u^i}, \qquad y_i = \frac{\partial y}{\partial u^i} \qquad i = 1, 2$$

and [-, -, -] denotes a scalar triple product (or determinant) of the bracketed vectors. Thus (1) becomes

$$D(u^1, u^2) = \left| \frac{-[r, r_1, r_2]}{r^3[y, y_1, y_2]} \right|. \tag{3}$$

The problem becomes one of differential geometry and may be stated in the following way. Given $D(u^1, u^2)$ can an explicit surface $r(u^1, u^2)$ be found satisfying equations (2) and (3)? A solution to this problem is developed under certain constraints in the following sections.

3. The tensor analysis

The analysis of the problem posed in generalized coordinates is best conducted for brevity and elegance using standard tensor notation, a brief note of which may be found in the appendix with further details in Lipschutz (1969) or Spain (1965).

The vectors y, y_1, y_2 are linearly independent and may be used as a basis for the vector field. Thus we may express

$$-r = qy + p^1 y_1 + p^2 y_2 = qy + p^i y_i \tag{4}$$

using the Einstein repeated index summation convention, where $q, p^i, i = 1, 2$, are coefficients to be determined. In fact p^i refers to the contravariant component of $-r$ referred to y_i.

A metric tensor $g_{ij} = y_i \cdot y_j$, $i, j = 1, 2$, is defined on the surface of the unit sphere $|y| = 1$.

The vectors y_i are tangential to the sphere so that $y \cdot y_i = 0$ and

$$r^2 = |r|^2 = q^2 + p^i p_i, \tag{5}$$

where the covariant components p_i and the contravariant components p^i are related (see appendix) by

$$p_i = g_{ij} p^j. \tag{6}$$

Equation (2) becomes

$$n = (q + r)y + p^i y_i. \tag{7}$$

Differentiating (4) with respect to u^i,

$$-r_i = qy_i + q_i y + \frac{\partial p^j}{\partial u^i} y_j + p^j y_{ij}. \tag{8}$$

The Gauss–Weingarten equations (see Lipschutz 1969) show that

$$y_{ij} = \Gamma^k_{ij} y_k - g_{ij} y, \tag{9}$$

where Γ^k_{ij} is a Christoffel symbol defined by

$$\Gamma^k_{ij} = \Gamma^k_{ji} = \frac{1}{2} g^{k\alpha}\left(\frac{\partial g_{j\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha}\right)$$

and $\|g^{\alpha\beta}\|$, $\|g_{\alpha\beta}\|$ are inverse matrices.

Combining (6), (8) and (9),

$$-\boldsymbol{r}_i = (q_i - p_i)\boldsymbol{y} + q\boldsymbol{y}_i + \left(\frac{\partial p^j}{\partial u^i} + p^k \Gamma^j_{ik}\right)\boldsymbol{y}_j. \tag{10}$$

But $\boldsymbol{r}_i$ is tangential to the reflector surface so that $\boldsymbol{n} \,.\, \boldsymbol{r}_i = 0$ and (7), (10) are used to yield

$$-\boldsymbol{n} \,.\, \boldsymbol{r}_i = 0 = (q+r)(q_i - p_i) + qp_i + p_j\left(\frac{\partial p^j}{\partial u^i} + p^k \Gamma^j_{ik}\right);$$

ie

$$0 = qq_i + r(q_i - p_i) + p_j p^j{}_i, \tag{11}$$

where $p^j{}_i$ is termed the *covariant derivative* of p^j with respect to u^i and defined by

$$p^j{}_i \equiv \frac{\partial p^j}{\partial u^i} + p^k \Gamma^j_{ik}.$$

The covariant differentiation of (5) with respect to u^i gives

$$rr_i = qq_i + p_j p^j{}_i \tag{12}$$

and so by combining (11), (12) we have

$$r(r_i + q_i - p_i) = 0.$$

Hence $p_i = q_i + r_i$, which is satisfied if we take

$$p = q + r. \tag{13}$$

Using (5) and (13)

$$p^2 - 2pq + q^2 = r^2 = q^2 + p^i p_i.$$

Therefore,

$$\begin{aligned} q &= \tfrac{1}{2}(p - p^{-1} p^i p_i) \\ r &= \tfrac{1}{2}(p + p^{-1} p^i p_i). \end{aligned} \tag{14}$$

Substituting (13), (14) into (4) we obtain

$$-\boldsymbol{r} = r\boldsymbol{y} + p^i \boldsymbol{V}_i \tag{15}$$

where

$$\boldsymbol{V}_i = \boldsymbol{y}_i - p^{-1} p_i \boldsymbol{y}. \tag{16}$$

From (11), (13)

$$q_i = p^{-1}(rp_i - p_j p^j{}_i),$$

and hence

$$r_i = p_i - q_i = p^{-1}(qp_i + p_j p^j{}_i).$$

But

$$\begin{aligned} -\boldsymbol{r}_i &= (q_i - p_i)\boldsymbol{y} + q\boldsymbol{y}_i + \boldsymbol{y}_j p^j{}_i \\ &= -p^{-1}(qp_i + p_j p^j{}_i)\boldsymbol{y} + q\boldsymbol{y}_i + \boldsymbol{y}_j p^j{}_i \\ &= q\boldsymbol{V}_i + p^j{}_i \boldsymbol{V}_j \\ &= (q\delta^j_i + p^j{}_i)\boldsymbol{V}_j. \end{aligned} \tag{17}$$

Hence

$$\begin{aligned} [\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2] &= -[r\boldsymbol{y} + p^i\boldsymbol{V}_i, (q\delta^j_1 + p^j{}_1)\boldsymbol{V}_j, (q\delta^j_2 + p^j{}_2)\boldsymbol{V}_j] \\ &= -r[\boldsymbol{y}, \boldsymbol{V}_1, \boldsymbol{V}_2]\det(p^j{}_i + q\delta^j_i) \\ &= -r[\boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2]\det(p^j{}_i + q\delta^j_i) \end{aligned} \tag{18}$$

since the components of $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ in the $\boldsymbol{y}$ direction do not contribute to the scalar triple product.

Substituting (18) into (3) we obtain the partial differential equation for $p(u^1, u^2)$ in the form

$$D(u^1, u^2) = \det(p^j{}_i + q\delta^j_i)/r^2, \tag{19}$$

where

$$p^j{}_i = \partial p^j/\partial u^i + p^k\Gamma^j_{ik}, \qquad q = \tfrac{1}{2}(p - p^{-1}p^ip_i), \qquad r = \tfrac{1}{2}(p + p^{-1}p^ip_i).$$

In the next section we examine it more closely in spherical polar coordinates.

4. Analysis in spherical polars

Spherical polar coordinates $u^1 = \theta$, $u^2 = \phi$ are taken on the unit sphere $|\boldsymbol{y}| = 1$ so that the components referred to Cartesian axes taken at 0 are

$$\begin{aligned} \boldsymbol{y} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \boldsymbol{y}_1 = \boldsymbol{y}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \boldsymbol{y}_2 = \boldsymbol{y}_\phi &= (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0). \end{aligned}$$

Then

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \sin^2\theta, \quad g^{11} = 1, \quad g^{12} = 0, \quad g^{22} = 1/\sin^2\theta$$

$$\Gamma^1_{11} = 0, \quad \Gamma^1_{12} = 0, \quad \Gamma^1_{22} = -\sin\theta\cos\theta, \quad \Gamma^2_{11} = 0, \quad \Gamma^2_{21} = \cot\theta, \quad \Gamma^2_{22} = 0$$

$$p^1 = g^{1j}p_j = p_1 = p_\theta, \qquad p^2 = g^{2j}p_j = p_\phi/\sin^2\theta, \qquad p^1{}_1 = p_{\theta\theta}$$

$$p^1{}_2 = p^2{}_1 = \frac{\partial p^2}{\partial\theta} + p^1\Gamma^2_{11} + p^2\Gamma^2_{12} = (p_{\theta\phi} - \cot\theta p_\phi)/\sin^2\theta$$

$$p^2{}_2 = \frac{\partial p^2}{\partial\phi} + p^1\Gamma^2_{21} + p^2\Gamma^2_{22} = (p_{\phi\phi} + \sin\theta\cos\theta p_\theta)/\sin^2\theta.$$

Thus (15) becomes

$$\boldsymbol{r} = -q\boldsymbol{y} - p_\theta\boldsymbol{y}_\theta - p_\phi\boldsymbol{y}_\phi/\sin^2\theta \tag{20}$$

and (19) becomes

$$r^2 \sin^2\theta D(\theta, \phi) = |(p_{\theta\theta}+q)(p_{\phi\phi}+\sin\theta\cos\theta p_\theta+q\sin^2\theta)-(p_{\theta\phi}-\cot\theta p_\phi)^2|,$$

where

$$q = \tfrac{1}{2}[p-p^{-1}(p_\theta^2+p_\phi^2/\sin^2\theta)], \qquad r = p-q.$$

It is evident therefore that the dependent variable $p(\theta, \phi)$ is positive and satisfies a second-order nonlinear partial differential equation of the form

$$p_{\theta\theta}p_{\phi\phi}-p_{\theta\phi}^2 = Ap_{\theta\theta}+2Bp_{\theta\phi}+Cp_{\phi\phi}+E\pm r^2\sin^2\theta D(\theta, \phi), \tag{21}$$

where A, B, C, E are functions of θ, ϕ, p, p_θ and p_ϕ. This equation is of the Monge–Ampére type, particular forms of which have been studied in detail by numerous authors in differential geometry (see for example Pogorelov 1964, 1973), but have not been previously used, we believe, in connection with reflector synthesis.

Any suitable solution of (21) satisfying certain boundary conditions to be imposed may be inserted into (20) to generate a reflector surface $\boldsymbol{r} = \boldsymbol{r}(\theta, \phi)$.

The Monge–Ampére form can be retained and slightly simplified by transforming the dependent variable by writing $p(\theta, \phi) = \exp(\sigma(\theta, \phi))$. Then σ satisfies the equation

$$\sigma_{\theta\theta}\sigma_{\phi\phi}-\sigma_{\theta\phi}^2 = a\sigma_{\theta\theta}+2b\sigma_{\theta\phi}+c\sigma_{\phi\phi}+d\pm eD(\theta, \phi), \tag{22}$$

where

$$\begin{aligned}
a &= -(\sigma_\phi^2+\sin\theta\cos\theta\sigma_\theta+q_{\mathrm{n}}\sin^2\theta)\\
b &= \sigma_\phi(\sigma_\theta-\cot\theta)\\
c &= -(\sigma_\theta^2+q_{\mathrm{n}})\\
d &= -(\sigma_\theta^2+q_{\mathrm{n}})(\sin\theta\cos\theta\sigma_\theta+q_{\mathrm{n}}\sin^2\theta)-\sigma_\phi^2(q_{\mathrm{n}}-\cot^2\theta+2\sigma_\theta\cot\theta)\\
e &= r_{\mathrm{n}}^2\sin^2\theta,
\end{aligned}$$

with

$$\begin{aligned}
r_{\mathrm{n}} &= r\,\mathrm{e}^{-\sigma} = \tfrac{1}{2}(1+\sigma_\theta^2+\sigma_\phi^2/\sin^2\theta)\\
q_{\mathrm{n}} &= q\,\mathrm{e}^{-\sigma} = \tfrac{1}{2}(1-\sigma_\theta^2-\sigma_\phi^2/\sin^2\theta)
\end{aligned}$$

as normalized forms of r, q respectively.

The form of equation (22) yields a number of points of interest about its solutions.

(1) The coefficients a, b, c, d, e are functions of θ, σ_θ and σ_ϕ but *not* of σ. Thus an arbitrary constant can be added to any solution and the equation will still be satisfied. This can be used to advantage in numerical solutions.

(2) If $D(\theta, \phi)$ is an even function of azimuthal angle ϕ then $\sigma(\theta, -\phi)$ satisfies the same equation as $\sigma(\theta, \phi)$. Hence if the boundary conditions have even symmetry in ϕ then the reflector will also exhibit this symmetry.

(3) Due to the coordinate system a singularity at $\theta = 0$ exists in some of the coefficients. An alternative form of rationalized coordinates can be used to avoid this pole and is presented in a subsequent paper.

(4) Assuming that $D(\theta, \phi)$ is positive and non-singular it is possible to show that a one-to-one mapping exists between points on the far-field sphere and points on the reflector surface. (This is a necessary requirement for a geometric-optics design to be a fair approximation to a diffraction-limited antenna.)

(5) The question of whether the equation is elliptic or hyperbolic is decided by the sign of the discriminant

$$Q = ac - b^2 + d \pm eD,$$

which after some simplification becomes

$$Q = \pm r_n^2 \sin^2\theta D. \tag{23}$$

Thus the equation is either elliptic if the upper sign is taken or hyperbolic with the lower sign assumed. Elliptic solutions for $\sigma(\theta, \phi)$ can produce reflectors possessing either *two* or *no* caustic surfaces. Hyperbolic solutions can yield reflectors possessing *one* caustic surface. Thus three distinct classes of reflector are possible.

(6) The question of conditions for existence of solutions for the Monge–Ampère equation is not considered in this paper. A good account of the literature in this field is given by Pogorelov (1964, 1973) but the Monge–Ampère forms he uses are more specialized than those given here. Thus he refers to *strongly elliptic* equations in which the coefficients are restricted by the conditions that $a\xi^2 + 2b\xi\eta + c\eta^2$ is positive definite and $d + eD > 0$, which are not in general satisfied by (22). Despite the shortcomings of existing theory numerical solutions have been obtained by us and have led to interesting reflector designs.

5. Boundary conditions for ϕ-even symmetric fields

In this paper we confine attention to elliptic forms of (22) and boundary conditions are developed which enable a successful approach to ϕ-even symmetric fields to be made.

The far-field cone of solid angle Ω intersects the far-field sphere R_∞ in a circle Γ. The axis of this cone is assumed to be the positive 0z axis of the far-field 0x, y, z coordinate system. The edge rays to Γ are defined by $\theta = \theta_{\Gamma}$, $0 \leqslant \phi \leqslant 2\pi$.

The reflector axes 0x', y, z' are obtained by a negative rotation ψ of the far-field axes about the common axis 0y. Thus we obtain a tilted feed configuration as shown in figure 2, where the reflector occupies a solid angle Ω' bounded by a right circular cone $\theta' = \theta'_{\Gamma}$, $0 \leqslant \phi' \leqslant 2\pi$.

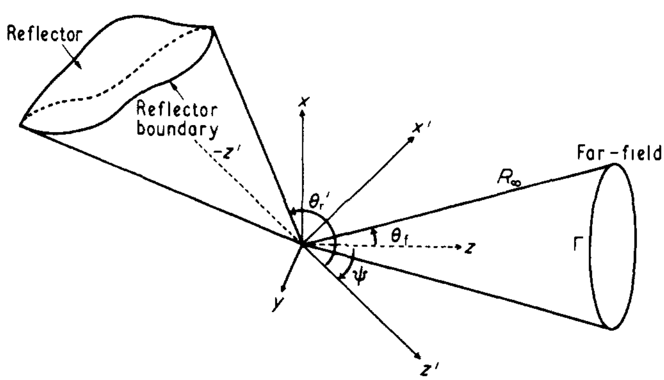

Figure 2. Respective coordinate axes for far-field and reflector with tilted feed configuration.

Boundary conditions are now obtained by the one-to-one mapping of points on the reflector boundary, which is assumed to be the boundary of a cross section, *not necessarily plane*, of the cone $\theta' = \theta'_r$, onto points on Γ. These conditions must be compatible with the energy conservation law

$$\int_0^{2\pi}\int_{\theta'_r}^{\pi} I(\theta', \phi') \sin\theta' \, d\theta' \, d\phi' = \int_0^{2\pi}\int_0^{\theta_f} G(\theta, \phi) \sin\theta \, d\theta \, d\phi, \tag{24}$$

which assumes that there is no folding of the reflector causing shadowing effects. This is also a necessary requirement to avoid blockage to incident or reflected rays.

The energy law effectively expresses θ'_r in terms of θ_f. For consider a simple case in which the source is an isotropic radiator over Ω' so that $I = I_0$, a constant over Ω'. Then

$$1 + \cos\theta'_r = \frac{1}{2\pi I_0}\int_0^{2\pi}\int_0^{\theta_f} G(\theta, \phi) \sin\theta \, d\theta \, d\phi.$$

An arbitrary point on the reflector has coordinates (x', y', z') referred to reflector axes or (x, y, z) referred to far-field axes, and if all coordinates are normalized by division by e^{σ} we have

$$x' = r_n \sin\theta' \cos\phi' = z \sin\psi + x \cos\psi \tag{25a}$$

$$y' = r_n \sin\theta' \sin\phi' = y \tag{25b}$$

$$z' = r_n \cos\theta' = z \cos\psi - x \sin\psi. \tag{26}$$

But from (20)

$$x = (\sigma_\phi \sin\phi/\sin\theta) - (q_n \sin\theta + \sigma_\theta \cos\theta) \cos\phi \tag{27a}$$

$$y = -(\sigma_\phi \cos\phi/\sin\theta) - (q_n \sin\theta + \sigma_\theta \cos\theta) \sin\phi \tag{27b}$$

$$z = \sigma_\theta \sin\theta - q_n \cos\theta, \tag{28}$$

where

$$q_n = \tfrac{1}{2}(1 - \sigma_\theta^2 - \sigma_\phi^2/\sin^2\theta), \qquad r_n = 1 - q_n.$$

When (26), (27*a*) and (28) are combined θ' is related to θ and the equation

$$A\sigma_\theta^2 + B\sigma_\phi^2 + C\sigma_\theta + D\sigma_\phi + E = 0, \tag{29}$$

where

$$A = B\sin^2\theta = \cos\theta' + \sin\theta \sin\psi \cos\phi - \cos\theta \cos\psi$$

$$C = -2\sin\theta\cos\psi - 2\cos\theta \sin\psi \cos\phi$$

$$D = 2\sin\psi \sin\phi/\sin\theta$$

$$E = \cos\theta' - \sin\theta \sin\psi \cos\phi + \cos\theta\cos\psi,$$

must be satisfied by the partial derivatives. The condition is quite general and does not depend on any symmetry property of the field. It must be satisfied in particular by the boundary values $\sigma_\theta(\theta_f, \phi)$, $\sigma_\phi(\theta_f, \phi)$.

It is clear that when (29) is used as a boundary condition the boundary mapping is not completely specified since the relationship $\phi' \to \phi$ is not defined.

For fields which are even functions of ϕ about the $0x$ axis it is reasonable to assume from a previous discussion that $\sigma(\theta, \phi)$ is also symmetric about the $0x$ axis. In fact we shall assume that $\sigma_\phi = 0$ when $y = 0$ (ie $\phi = 0$ or π). The corresponding values of σ_θ are given by (29) as the solution of the quadratic

$$A'\sigma_\theta^2 + C'\sigma_\theta + E' = 0 \qquad \phi = 0, \pi, \tag{30}$$

where

$$A' = \cos\theta' - \cos(\theta \pm \psi)$$

$$C' = -2\sin(\theta \pm \psi)$$

$$E' = \cos\theta' + \cos(\theta \pm \psi),$$

in which the choice of the upper sign refers to $\phi = 0$ and the lower sign to $\phi = \pi$. Thus we obtain

$$\sigma_\theta(\theta, 0) = \frac{\sin(\theta + \psi) \pm \sin\theta'}{\cos\theta' - \cos(\theta + \psi)} = \cot\tfrac{1}{2}(\theta + \psi \mp \theta'), \tag{31a}$$

$$\sigma_\theta(\theta, \pi) = \frac{\sin(\theta - \psi) \pm \sin\theta'}{\cos\theta' - \cos(\theta - \psi)} = \cot\tfrac{1}{2}(\theta - \psi \mp \theta'). \tag{31b}$$

Consideration of equations (31) with (25) and (27) shows that when the upper signs are assumed then $\phi' = 0$ when $\phi = 0$ and $\phi' = \pi$ when $\phi = \pi$, ie edge rays link 'top' of reflector with 'top' of far-field circle Γ and 'bottom' of reflector with 'bottom' of Γ. Alternatively when the lower signs are chosen, $\phi' = 0$ when $\phi = \pi$ and $\phi' = \pi$ when $\phi = 0$ indicating a 'top–bottom' and 'bottom–top' link respectively. For either choice of signs only one of the pair (31a), (31b) needs to be considered, the other is satisfied automatically when $\sigma_\phi = 0$.

Boundary conditions for the azimuthally symmetric field problem suggested by the above discussion are summarized below:

(i) Energy conservation equation (24) which relates θ_r' to θ_f
(ii) $A\sigma_\theta^2 + B\sigma_\phi^2 + C\sigma_\theta + D\sigma_\phi + E = 0$ for $\theta = \theta_f$, $\theta' = \theta_r'$ and all ϕ in $(0, \pi)$
(iii) $\sigma_\phi(\theta, 0) = \sigma_\phi(\theta, \pi) = 0$ for all θ in $(0, \theta_f)$
(iv) $\sigma_\theta(\theta_f, 0) = \cot\tfrac{1}{2}(\theta_f + \psi \mp \theta_r')$
(v) $\sigma(0, \phi) = 0$.

Condition (iv) is not independent but consistent with (ii) and (iii). The condition (v) is introduced because the Monge–Ampère equation and the boundary conditions involve only partial derivatives of σ and not σ itself so that without (v) σ is only determined to within an arbitrary constant.

It has been found that (i)–(v) are sufficient to enable useful solutions to be found numerically. These solutions are inserted into (20) to obtain the corresponding reflector surfaces. The results are to be presented in a subsequent paper.

6. Existence and uniqueness

The lack of relevant theorems referred to in §4 implies that any conclusions based upon numerical work must be purely speculative and further work in pure mathematics is necessary to substantiate conditions for existence and uniqueness. Nevertheless some implications can be deduced by examining the boundary conditions more closely.

The quadratic (29) on the boundary represents the equation of a circle, for if

$$\xi = \sigma_\theta(\theta_f, \phi), \qquad \eta = \sigma_\phi(\theta_f, \phi)/\sin\theta_f$$

then

$$A(\xi^2+\eta^2)+C\xi+\eta D\sin\theta_f+E=0. \tag{32}$$

The centre of the circle is (ξ_0, η_0) and the radius is ρ where

$$\xi_0 = -\frac{C}{2A} = -\frac{\sin\theta_f\cos\psi+\cos\theta_f\sin\psi\cos\phi}{\cos\theta_f\cos\psi-\cos\theta_r'-\sin\theta_f\sin\psi\cos\phi}$$

$$\eta_0 = -\frac{D\sin\theta_f}{2A} = \frac{\sin\psi\sin\phi}{\cos\theta_f\cos\psi-\cos\theta_r'-\sin\theta_f\sin\psi\cos\phi}$$

$$\rho = \left(\frac{C^2+D^2\sin^2\theta_f-4AE}{4A^2}\right)^{\frac{1}{2}} = \frac{\sin\theta_r'}{|\cos\theta_f\cos\psi-\cos\theta_r'-\sin\theta_f\sin\psi\cos\phi|}.$$

The circle is fixed in general by specifying values for the parameters $\theta_f, \theta_r', \psi, \phi$ and for each set of parameter values the boundary values ξ, η must lie on the perimeter of the corresponding circle. Thus permitted values of ξ, η are bounded by $|\xi-\xi_0| \leqslant \rho$, $|\eta-\eta_0| \leqslant \rho$.

To illustrate the situation a series of these circles is drawn in figure 3 for values of ϕ in the range $(0, \pi)$ and for the particular values $\theta_f = \pi/3$, $\theta_r' = 2\pi/3$ and $\psi = \pi/6$. It should be noted that the mirror image in the $\eta = 0$ axis is obtained, though not plotted, when ϕ takes values in the range $(\pi, 2\pi)$. Superimposed is the circle $\xi^2+\eta^2 = 1$ which corresponds to $r_n = 1$ and $q_n = 0$. Hence boundary values which fall within this circle have $r_n < 1$ and $q_n > 0$ while those outside have $r_n > 1$ and $q_n < 0$. The origin $\xi = 0$, $\eta = 0$ corresponds to $r_n = \frac{1}{2}$ which is associated with a spherical reflector. The four values of ξ for which $\eta = 0$ on the circles $\phi = 0$ and π are of course consistent with condition (iv) of § 5.

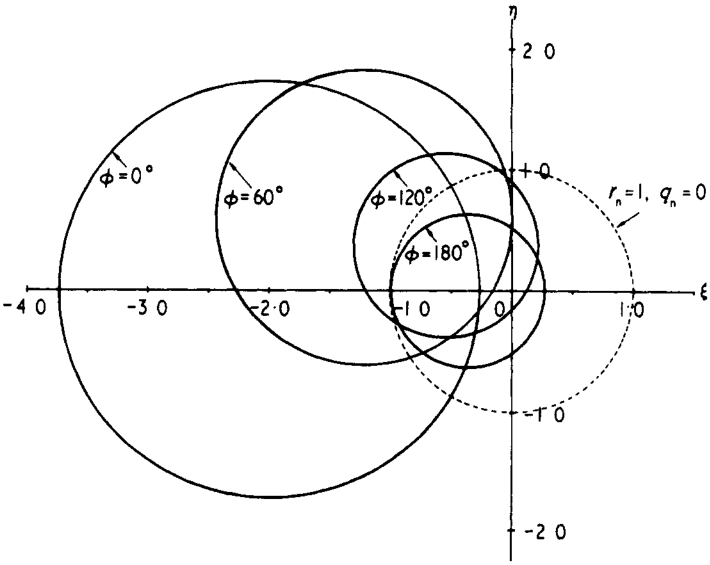

Figure 3. Permissible boundary value loci for various values of ϕ in $(0, \pi)$ with $\theta_f = \pi/3$, $\theta_r' = 2\pi/3$ and $\psi = \pi/6$, where $\xi = \sigma_\theta(\theta_f, \phi)$, $\eta = \sigma_\phi(\theta_f, \phi)/\sin\theta_f$. The dotted circle corresponds to $r_n = 1, q_n = 0$.

The situation is less complicated when $\psi = 0$, for then (ξ, η) lies on the circle defined by

$$\xi_0 = -\frac{\sin \theta_f}{\cos \theta_f - \cos \theta_r'}, \qquad \eta_0 = 0, \qquad \rho = \frac{\sin \theta_r'}{|\cos \theta_f - \cos \theta_r'|}$$

for all ϕ. At $\phi = 0$ the point $(\xi, \eta) = (\xi_0 \pm \rho, 0)$ consistent with condition (iv), and as ϕ increases from 0 to π (ξ, η) takes values consistent with the solution for $\sigma(\theta_f, \phi)$, finally returning to its starting value when $\phi = \pi$.

7. Conclusions

A new method has been proposed in which reflector surfaces may be synthesised to produce a given far-field radiation pattern when illuminated by a point source. The method, based upon the geometric-optics approximation, offers more flexibility to the reflector designer in that far-field patterns are allowed to vary in a prescribed way with two coordinates over a solid angle as compared with one variable in current systematic methods.

The foundations of the theory are presented in this paper and it is shown that the solution of the problem hinges on the ability to solve a nonlinear second-order partial differential equation, namely the Monge–Ampere equation, subject to certain nonlinear boundary conditions. Theorems of existence and uniqueness currently available are too restrictive to be applied to the generalized equations presented here.

The boundary conditions are examined in detail for even azimuthal symmetric fields for elliptic forms of the Monge–Ampère equation and it is believed that at least two solutions are possible for this case. In fact this problem has been successfully programmed for a digital computer and numerical results together with further developments of the theory will be presented in subsequent papers.

The theory presented here should be applicable to a wide variety of problems in optics and microwaves in which a shaped beam is required. One particular advantage to the microwave antenna designer is that only the amplitude and *not* the phase of the far-field needs to be specified.

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Appendix. Tensor notation

The equation of a surface may be written $\boldsymbol{r} = \boldsymbol{r}(u^1, u^2)$ where u^1, u^2 are generalized curvilinear coordinates. The vectors $\boldsymbol{r}_1 = \partial \boldsymbol{r}/\partial u^1$, $\boldsymbol{r}_2 = \partial \boldsymbol{r}/\partial u^2$ are linear independent vectors in the tangent plane at the point $P(u^1, u^2)$. An arbitrary vector in this plane may be expressed as a linear combination of the two:

$$\boldsymbol{a} = a^i \boldsymbol{r}_i = a^1 \boldsymbol{r}_1 + a^2 \boldsymbol{r}_2$$

where a^1, a^2 are the *contravariant components* of $\boldsymbol{a}$ referred to $\boldsymbol{r}_i$, $i = 1, 2$. We define $g_{ij} = g_{ji} = \boldsymbol{r}_i \cdot \boldsymbol{r}_j$, $i, j = 1, 2$, to be the metric tensor in the tangent plane and g^{ij} a tensor such that

$$g_{ik}g^{kj} = \delta_i^j = \begin{matrix} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{matrix}$$

Then $\|g^{ij}\|$ is the inverse matrix of $\|g_{ij}\|$.

Defining a reciprocal basis $\boldsymbol{r}^1, \boldsymbol{r}^2$ such that $\boldsymbol{r}_i \cdot \boldsymbol{r}^j = \delta_i^j$, $i, j = 1, 2$, then

$$\boldsymbol{a} = a_i \boldsymbol{r}^i = a_1 \boldsymbol{r}^1 + a_2 \boldsymbol{r}^2,$$

where a_1, a_2 are the *covariant components* of $\boldsymbol{a}$ referred to $\boldsymbol{r}^i$.

The relationship between a_i and a^i is given by

$$a_j = a_i \delta_i^j = a_i \boldsymbol{r}^i \cdot \boldsymbol{r}_j = \boldsymbol{a} \cdot \boldsymbol{r}_j = a^i \boldsymbol{r}_i \cdot \boldsymbol{r}_j = g_{ij} a^i.$$

Similarly $a^j = g^{ij} a_i$.

References

Lipschutz M M 1969 *Differential Geometry* (New York: McGraw-Hill, Schaum's Outline Series)
Norris A P and Westcott B S 1974 *Electron Lett.* **15** 322–3
Pogorelov A V 1964 *Monge–Ampère Equations of Elliptic Type* (Groningen: Noordhoff, Trans. from Russian)
—— 1973 *Extrinsic Geometry of Convex Surfaces* (Jerusalem: IPST, Trans. from Russian)
Silver S 1949 *Microwave Antenna Theory and Design* (New York: McGraw-Hill, Radiation Lab. Series 12) Chap 13
Spain B 1965 *Vector Analysis* (London: Van Nostrand, New University Mathematics Series)